Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

CONJUGATE PLANAR NETS WITH EQUAL INVARIANTS.

BY LUTHER PFAHLER EISENHART.

Koenigs* has shown by geometrical considerations that the perspectives of the asymptotic lines on a surface from a point on a plane form a conjugate net with equal invariants; that is, the homogeneous coördinates y_1, y_2, y_3, of the plane, expressed in terms of parameters u and v referring to the planar net, are solutions of a system of equations of the form

$$\begin{aligned} y_{uu} &= ay_u + by_v + cy, \\ y_{uv} &= (\log \sigma)_v y_u + (\log \sigma)_u y_v + c'y, \\ y_{vv} &= a''y_u + b''y_v + c''y, \end{aligned} \tag{1}$$

where in general the coefficients are functions of u and v. Koenigs observed, conversely, that such a planar net may be regarded as the perspective of the asymptotic lines on a surface. He stated that the surface can be found by quadratures, but did not give a proof. It is the purpose of this note to establish this converse theorem, at the same time giving the quadratures involved in the determination of the surface. It is an interesting fact that the formulas of Lelieuvre arise as a special case of the problem.

In order that equations (1) shall be completely integrable, and thus admit three linearly independent solutions, the coefficients must satisfy six conditions obtained from the integrability requirements

$$(y_{uu})_v = (y_{uv})_u, \qquad (y_{uv})_v = (y_{vv})_u.$$

We shall not write down these conditions, but merely remark that one of them is $a_v = b_u$, which may be replaced by

$$a = \frac{\partial \log \rho}{\partial u}, \qquad b = \frac{\partial \log \rho}{\partial v}, \tag{2}$$

where ρ is thus defined.

It is well known† that when the first and third of equations (1) admit four linearly independent solutions, they are the homogeneous point coördinates of a surface upon which the parametric curves are the asymp-

---

[*] Comptes Rendus, vol. 114 (1892), p. 55. Recently an analytic proof has been given by A. L. Nelson, Rendiconti di Palermo, vol. 42 (1916).

[†] Darboux, Leçons, second edition, vol. 1, p. 205.

totic lines. There are four conditions to be satisfied by the coefficients of two such equations in order that they shall admit four linearly independent solutions. These conditions follow from the requirement that the equation

$$(y_{uu})_{vv} = (y_{vv})_{uu}$$

shall be satisfied identically. One of them is equivalent to (2), and the other three are satisfied when the system (1) is completely integrable. Hence when the system (1) is completely integrable, there exists a solution of the first and third of equations (1) linearly independent of the three linearly independent solutions of the system. Consequently the parametric asymptotic net on the surface whose point coördinates are these four solutions, is perspective with the planar net, defined by (1), from the vertex of the tetrahedron of reference not on the plane. Hence we have established the converse of the theorem of Koenigs. But we are interested primarily in showing how the fourth solution of the first and third of equations (1) can be found by three quadratures, when three linearly independent solutions of the system (1) are known. The remainder of the paper is concerned with this problem and certain corollaries.

3. It is desirable to recall the method of reducing to a quadrature the solution of an ordinary linear differential equation of order n when $n - 1$ linearly independent solutions are known.* We take the case $n = 4$ (as this applies to the problem in hand), and write the equation

$$\frac{d^4y}{dx^4} + \alpha\frac{d^3y}{dx^3} + \beta\frac{d^2y}{dx^2} + \gamma\frac{dy}{dx} + \delta y = 0. \tag{3}$$

Let y_1, y_2, y_3 be three linearly independent solutions and write

$$y = C_1y_1 + C_2y_2 + C_3y_3, \tag{4}$$

where C_i are functions of x subject to the conditions

$$\sum_{i=1}^{3} y_i\frac{dC_i}{dx_i} = 0, \qquad \sum_{i=1}^{3}\frac{dy_i}{dx}\frac{dC_i}{dx} = 0. \tag{5}$$

When we require that y as given by (4) shall satisfy equation (3), we get

$$\sum_{i=1}^{3}\left(2\frac{dC_i}{dx}\frac{d^3y_i}{dx^3} + \frac{d^2C_i}{dx^2}\frac{d^2y_i}{dx^2} + \alpha\frac{dC_i}{dx}\frac{d^2y_i}{dx^2}\right) = 0. \tag{6}$$

In order to give this equation a suitable form, we make use of the determinant

---

* Forsyth, Differential Equations, third edition, pp. 128–130.

$$(7) \qquad \Delta = \begin{vmatrix} \frac{d^2y_1}{dx^2} & \frac{d^2y_2}{dx^2} & \frac{d^2y_3}{dx^2} \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \frac{dy_3}{dx} \\ y_1 & y_2 & y_3 \end{vmatrix},$$

and indicate by Δ_i the cofactor of d^2y_i/dx^2. From (5) we get

$$\frac{\frac{dC_1}{dx}}{\Delta_1} = \frac{\frac{dC_2}{dx}}{\Delta_2} = \frac{\frac{dC_3}{dx}}{\Delta_3} = z,$$

z being thus defined. Hence we have

$$\frac{d^2C_i}{dx^2} = \Delta_i\frac{dz}{dx} + z\frac{d\Delta_i}{dx}.$$

When these values are substituted in (6), the resulting equation is reducible to the form

$$\frac{d}{dx}\log z\Delta^2 = -\alpha.$$

Consequently

$$z = \frac{A}{\Delta^2}e^{-\int\alpha dx}, \qquad C_i = A\int\frac{\Delta_i}{\Delta^2}e^{-\int\alpha dx}dx,$$

where A is an arbitrary constant. The latter can in all generality be taken equal to unity, so that the integral (4) is

$$(8) \qquad y = \Sigma y_i\int\frac{\Delta_i}{\Delta^2}e^{-\int\alpha dx}dx,$$

which evidently is linearly independent of y_1, y_2, and y_3.

4. If the first of equations (1) be differentiated twice with respect to u, the resulting equation is reducible by means of the first and third of (1) to

$$(9) \qquad y_{uuuu} - 2(\log b\rho)_u y_{uuu} + Py_{uu} + Qy_u + Ry = 0,$$

where P, Q, R are determinate functions of u and v whose form is unessential to the subsequent study.

For the present we look upon equation (9) as an ordinary differential equation with v appearing as a parameter in the coefficients, and apply the method of § 3, on the supposition that we know the planar net, that is the homogeneous coördinates y_1, y_2, y_3 satisfying (1), which are three linearly independent solutions of (9). Now Δ has the form

$$\Delta = \begin{vmatrix} \frac{\partial^2 y_1}{\partial u^2} & \frac{\partial^2 y_2}{\partial u^2} & \frac{\partial y_3}{\partial u^2} \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_2}{\partial u} & \frac{\partial y_3}{\partial u} \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

If this equation be differentiated with respect to u, the result is reducible to

$$\frac{\partial \Delta}{\partial u} = \Delta \frac{\partial}{\partial u} \log (b\sigma\rho).$$

Hence for the present case the integral (8) becomes

$$y = \sum_{i=1}^{3} y_i \int \frac{\Delta_i}{\sigma^2} du.$$

If we had proceeded in a similar way by differentiating twice with respect to v the third of equations (1), we should have obtained the integral

$$y = \sum_{i=1}^{3} y_i \int \frac{\Delta_i'}{\sigma^2} dv,$$

where Δ_i' are the cofactors of the terms of the first row of

$$\Delta' = \begin{vmatrix} \frac{\partial^2 y_1}{\partial v^2} & \frac{\partial^2 y_2}{\partial v^2} & \frac{\partial^2 y_3}{\partial v^2} \\ \frac{\partial y_1}{\partial v} & \frac{\partial y_2}{\partial v} & \frac{\partial y_3}{\partial v} \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

In order to coördinate these results, we observe that for $i = 1, 2, 3$; $j = 1, 2, 3$, but $i \neq j$,

$$\left(\frac{y_i y_{ju} - y_j y_{iu}}{\sigma^2}\right)_v + \left(\frac{y_i y_{jv} - y_j y_{iv}}{\sigma^2}\right)_u = 0.$$

Hence looking upon u and v as independent variables, we note that a fourth integral of the first and third of equations (1) is given by

$$y_4 = x_1 y_1 + x_2 y_2 + x_3 y_3, \tag{10}$$

where the functions x_i are given by the quadratures

$$x_{1u} = \frac{1}{\sigma^2}(y_2 y_{3u} - y_3 y_{2u}), \qquad x_{1v} = -\frac{1}{\sigma^2}(y_2 y_{3v} - y_3 y_{2v}), \tag{11}$$

and similar expressions obtained by permuting the subscripts cyclically.

5. Since any linear expression with constant coefficients of y_1, y_2, y_3, y_4 is likewise a solution of the first and third of equations (1), there are ∞^4 asymptotic nets which are perspective with the planar net from the vertex of the coördinate tetrahedron not in the plane of the given net.

6. From (11) we have by differentiation

$$(12)\qquad \begin{aligned} x_{iuu} &= \left(a - \frac{2\sigma_u}{\sigma}\right)x_{iu} - bx_{iv}, \\ x_{ivv} &= -a''x_{iu} + \left(b'' - \frac{2\sigma_v}{\sigma}\right)x_{iv}. \end{aligned}$$

Hence x_1, x_2, x_3, 1 are the homogeneous point coördinates of a surface X referred to its asymptotic lines.

From (10) it follows that X is the polar of the surface Y with the point coördinates y_1, y_2, y_3, y_4, with respect to the quadric

$$z_1^2 + z_2^2 + z_3^2 - z_4^2 = 0.$$

The functions y_i may also be looked upon as tangential coördinates of X. Consequently y_1, y_2, y_3 are the tangential coördinates of the planar net enveloped by the lines of intersection of the tangent planes to X and the plane $z_4 = 0$.

7. If we put

$$\nu_i = \frac{y_i}{\sigma},$$

equation (10) becomes

$$\nu_4 = x_1\nu_1 + x_2\nu_2 + x_3\nu_3,$$

and (11) may be replaced by

$$(13)\qquad x_{1u} = \nu_2\nu_{3u} - \nu_3\nu_{2u}, \qquad x_{1v} = -(\nu_2\nu_{3v} - \nu_3\nu_{2v}).$$

If now we take $z_4 = 0$ as the plane at infinity, and the other edges of the coördinate tetrahedron are mutually perpendicular, the functions ν_1, ν_2, ν_3 are proportional to the direction-cosines of the normal to X, and equations (13) are the well-known formulas of Lelieuvre.*

8. In order that the coördinates of the given planar net may be cartesian, the functions c in equations (1) must be zero. In this case we put $y_3 = 1$ in (10) and (11), so that the given net is the orthogonal projection of the asymptotic parametric net on the surface whose cartesian coördinates are y_1, y_2, y_4.

---

* Cf. the author's Differential Geometry, p. 193.

PRINCETON UNIVERSITY.